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THE MODELING OF DISTRIBUTED RC NETWORKS

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THE MODELING OF DISTRIBUTED RC NETWORKS

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ABSTRACT

A method for modeling the general three terminal distributed network is presented here. Two models to approximate the transmission parameters of a section of the distributed line are introduced. By using an iterative matrix multiplication process, the two models are able to approximate the admittance parameters of the distributed network. The values for the parameters as obtained from the models are compared with the theoretical values for the cases of the uniform and exponentially tapered lines. One of the models is shown to give better overall accuracy than the others.

I. INTRODUCTION

With the advent of micro-electronics, distributed RC networks are becoming increasingly important in electrical engineering. When the engineer is called upon to analyze a circuit which may contain distributed networks, he is confronted with two formidable obstacles. First, immittances associated with distributed RC networks will, in general, involve transcendental functions. This makes application of standard network analysis techniques difficult, to say the least. Second, analytical expressions for these immittances are available for only a few distributed networks and particular geometries, and these expressions are completely different for each particular geometry. For example, the admittance parameters associated with a distributed network with an exponential taper have a form entirely different from the admittance parameters of a distributed network with a linear taper.[1] The use of analysis techniques which employ the digital computer allow the engineer to surmount the numerical difficulties associated with the analysis of networks which contain distributed RC networks, and the use of a modeling technique to be presented here allow him to perform this analysis regardless of the geometrical configuration of the distributed networks.

Two models for the three terminal distributed network will be presented in this report. These models employ an iterative computation technique to approximate the admittance parameters of the distributed network. The models may be applied to any three terminal distributed netowrk, regardless of the geometrical configuration of that network. It will be shown that they give an accurate approximation to the admittance parameters of uniform distributed networks and distributed networks with an exponential taper. Finally, certain conclusions will be reached regarding the accuracy of each of the two models, and one of the models will be shown to give the more accurate overall approximation.

II. PRESENTATION OF THE MODELS

In this section two models are presented which may be used to approximate the admittance parameters of a three terminal distributed network at some specified frequency. Both of these models utilize the same basic concept. The distributed network to be analyzed is divided into a number of elemental sections, and the resistance and capacitance associated with each section is specified. First the transmission parameters of an elemental section are approximated at the frequency of interest, then an iterative procedure is used to obtain the transmission parameters of the distributed network. The admittance parameters of the network are then derived from the transmission parameters. The inaccuracies that arise in the models can be made quite small by dividing the network into a sufficient number of elemental sections. If a digital computer is used in the iterative computation process, the number of sections may easily be made large enough so that the approximation is very accurate.

The Difference Equation Approximation

The approximation of the transmission parameters of an elemental section by a difference equation technique will be considered first. A differential section of the distributed network may be approximated by the circuit shown in Fig. 2.1. If we neglect terms involving second order differentials, we have for sinusoidal steady state conditions:

$$\Delta i = -j\omega c\Delta x\ (v + \Delta v) \cong -j\omega cv\Delta x \tag{2.1}$$

$$\Delta v = -ir\Delta x \tag{2.2}$$

where r and c are respectively resistance and capacitance per unit length, ω is the angular frequency, and v, i, Δv, and Δi are phasors, representing the voltages and currents. We may now represent the voltages and currents at the ports of this differential section by the matrix equation

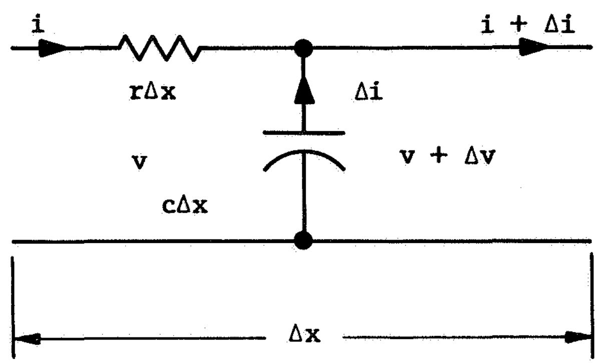

Fig. 2.1 Equivalent Circuit for a Differential Section of the RC Distributed Network

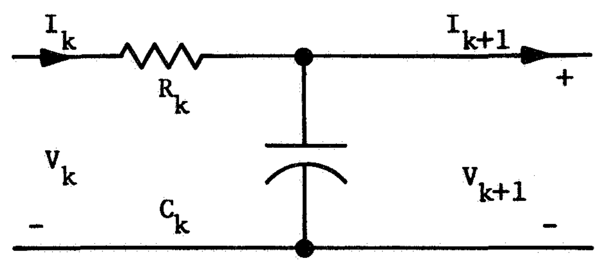

Fig. 2.2 Equivalent Circuit for an Elemental Section of the RC Distributed Network

$$\begin{bmatrix} v + \Delta v \\ \\ i + \Delta i \end{bmatrix} = \begin{bmatrix} 1 & -r\Delta x \\ \\ -j\omega c\Delta x & 1 \end{bmatrix} \begin{bmatrix} v \\ \\ i \end{bmatrix} \tag{2.3}$$

Equation (2.3) may be inverted to obtain:

$$\begin{bmatrix} v \\ \\ i \end{bmatrix} = \begin{bmatrix} 1 & r\Delta x \\ \\ j\omega c\Delta x & 1 \end{bmatrix} \begin{bmatrix} v + \Delta v \\ \\ i + \Delta i \end{bmatrix} \tag{2.4}$$

Again, terms involving second order differentials have been neglected.

Now let us suppose we have a distributed network divided into a number of elemental sections. The kth section may be approximated by the circuit shown in Fig. 2.2. We may now relate the port voltages and currents by an equation analogous to (2.4). This is

$$\begin{bmatrix} V_k \\ \\ I_k \end{bmatrix} = \begin{bmatrix} 1 & R_k \\ \\ j\omega C_k & 1 \end{bmatrix} \begin{bmatrix} V_{k+1} \\ \\ I_{k+1} \end{bmatrix} \tag{2.5}$$

where we again assume sinusoidal steady state conditions, and where ω is the angular frequency. R_k is the mean value of resistance per unit length of the section, multiplied by the length of the section. Similarly C_k is the mean value of the capacitance per unit length multiplied by the length of the section. I_k, I_{k+1}, V_k and V_{k+1} are phasors representing the voltages and currents.

The matrix of (2.5) is the transmission matrix of the kth elemental section. If the transmission matrices of each of the elemental sections of the line are multiplied together in order, the transmission matrix for the entire distributed network is approximated. The admittance parameters of the network are readily obtained from the transmission parameters.

A model equivalent to the Difference Equation Model has been reported in the literature.[2] As we shall see in this report, the model to be discussed next, namely, the Lumped Element Model provides better general results than the Difference Equation Model.

Lumped Element Approximations

A different approximation for the admittance parameters is obtained if we consider R_k and C_k in the network shown in Fig. 2.2 as elements in an ordinary lumped RC network and obtain a set of equations analogous to (2.5). Thus we find

$$V_{k+1} = V_k - I_k R_k \tag{2.6}$$

$$I_{k+1} = I_k - V_{k+1}\,(j\omega C_k) \tag{2.7}$$

substituting (2.6) for V_{k+1} into (2.7) results in:

$$I_{k+1} = I_k(1 + j\omega C_k R_k) - V_k(j\omega C_k) \tag{2.8}$$

Using matrix format we may write (2.6) and (2.8) as

$$\begin{bmatrix} V_{k+1} \\ I_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & -R_k \\ -j\omega C_k & 1 + j\omega C_k R_k \end{bmatrix} \begin{bmatrix} V_k \\ I_k \end{bmatrix} \tag{2.9}$$

Inverting the matrix of (2.9) to obtain the transmission parameters of the elemental section we have:

$$\begin{bmatrix} V_k \\ I_k \end{bmatrix} = \begin{bmatrix} 1 + j\omega C_k R_k & R_k \\ j\omega C_k & 1 \end{bmatrix} \begin{bmatrix} V_{k+1} \\ I_{k+1} \end{bmatrix} \tag{2.10}$$

The computation of the admittance parameters from (2.10) is easily accomplished.

In this section we have presented two models which may be used to provide an approximation to the admittance parameters of a distributed RC network. The two models differ only in the method used to approximate the transmission parameters of an elemental section of the network. The accuracy of the approximation depends upon the number of elemental sections into which the network is divided, the accuracy improving as the number of sections is increased.

III. RESULTS WITH A UNIFORM DISTRIBUTED NETWORK

In this section, we compare the results for the admittance parameters of the uniform distributed network with the theoretical values. A performance criterion is introduced to compare the accuracies of the models. Certain patterns in the approximation error associated with the models are observed.

Theoretical Values of the Admittance Parameters

Although analytical expressions for the admittance parameters of an arbitrarily tapered distributed RC network are not available in closed form, an exact representation for the admittance parameters of a distributed network with an exponential taper under conditions of sinusoidal steady state excitation has been obtained. An exponential network is characterized by the relations

$$r(x) = r_o\, e^{\alpha x} \tag{3.1}$$

$$c(x) = c_o\, e^{-\alpha x} \tag{3.2}$$

where r(x) and c(x) are the resistance and capacitance per unit length at some point along the line. Kaufman and Garrett have derived the admittance parameters of the exponential network used as a two-port.[1] If we define the total length of the network as L, the admittance parameters are given by

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \frac{1}{r_o L} \begin{bmatrix} \dfrac{\theta}{\tanh\,\theta} - \dfrac{\alpha L}{2} & \dfrac{-\theta e^{-\alpha L/2}}{\sinh\,\theta} \\ \dfrac{-\theta e^{-\alpha L/2}}{\sinh\,\theta} & e^{-\alpha L}\left[\dfrac{\theta}{\tanh\,\theta} + \dfrac{\alpha L}{2}\right] \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \tag{3.3}$$

where $\theta = \left(\frac{\alpha^2 L^2}{4} + j\omega r_o c_o L^2\right)^{1/2}$

For the case of the uniform network ($\alpha = 0$), (3.3) reduces to

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \frac{1}{R} \begin{bmatrix} \dfrac{\theta}{\tanh\,\theta} & \dfrac{-\theta}{\sinh\,\theta} \\ \dfrac{-\theta}{\sinh\,\theta} & \dfrac{\theta}{\tanh\,\theta} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \tag{3.4}$$

where $\theta = (j\omega RC)^{1/2}$ and R and C are respectively the total resistance and capacitance of the network.

It should be noted that in (3.3) the magnitude of the admittance parameters varies inversely with the resistance $r_o L$ when the quantities $\frac{\alpha L}{2}$ and $\omega r_o c_o L^2$ remain constant, and that the parameters depend only upon the product $\omega r_o c_o L^2$ when r_o and $\frac{\alpha L}{2}$ are held constant. Thus, when $\frac{\alpha L}{2}$ remains constant, the magnitude and frequency normalization characteristics resemble those of a lumped RC network. The parameter $\frac{\alpha L}{2}$ is defined as the degree of taper of the network.[1]

The comparison of results using the two models for the RC distributed network is simplified if the comparison is based on normalized values for the distributed network parameters. If the validity of the models is investigated for unity values of r_o, c_o and L, the conclusions reached will apply for any exponential network with the same degree of taper. In the case of the uniform distributed network, the equivalent conditions are unity values for R and C. All results that follow are given for normalized distributed network parameters.

Results Comparing the Accuracy of Each of the Parameters

A computer program was developed to compare the results for the admittance parameters of the uniform distributed network using both the Difference Equation Model and the Lumped Element Model with the theoretical values of the parameters obtained from (3.3). The results comparing the theoretical values of the y_{11} and y_{12} parameters with the values obtained using the models are shown in the graphs of Fig. 3.1 through Fig. 3.4. Both of the models used 40 elemental sections in the approximation. The graphs show that both models approximate the network to exactly the same degree of accuracy at low frequencies, but that the Difference Equation Model gives better results for y_{11} at high frequencies, and the Lumped Element Model gives better results for y_{12} at high frequencies. It was found that the results for the y_{21} parameter were significantly better using the Lumped Element Model, and that results for the y_{22} parameter were better using the Difference

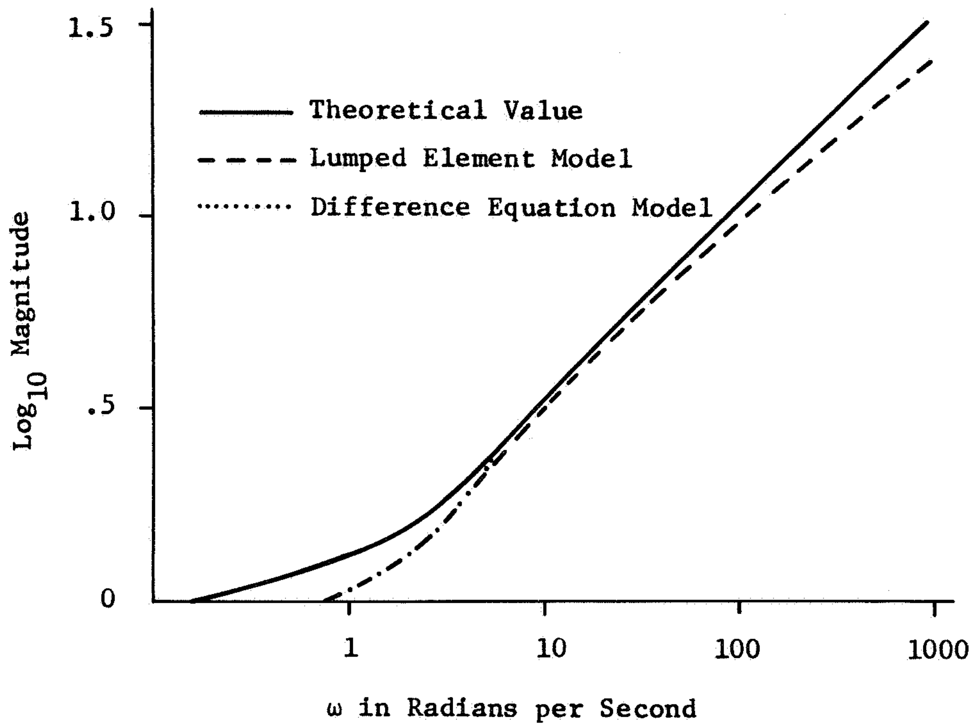

Fig. 3.1. Magnitude of the y_{11} Parameter

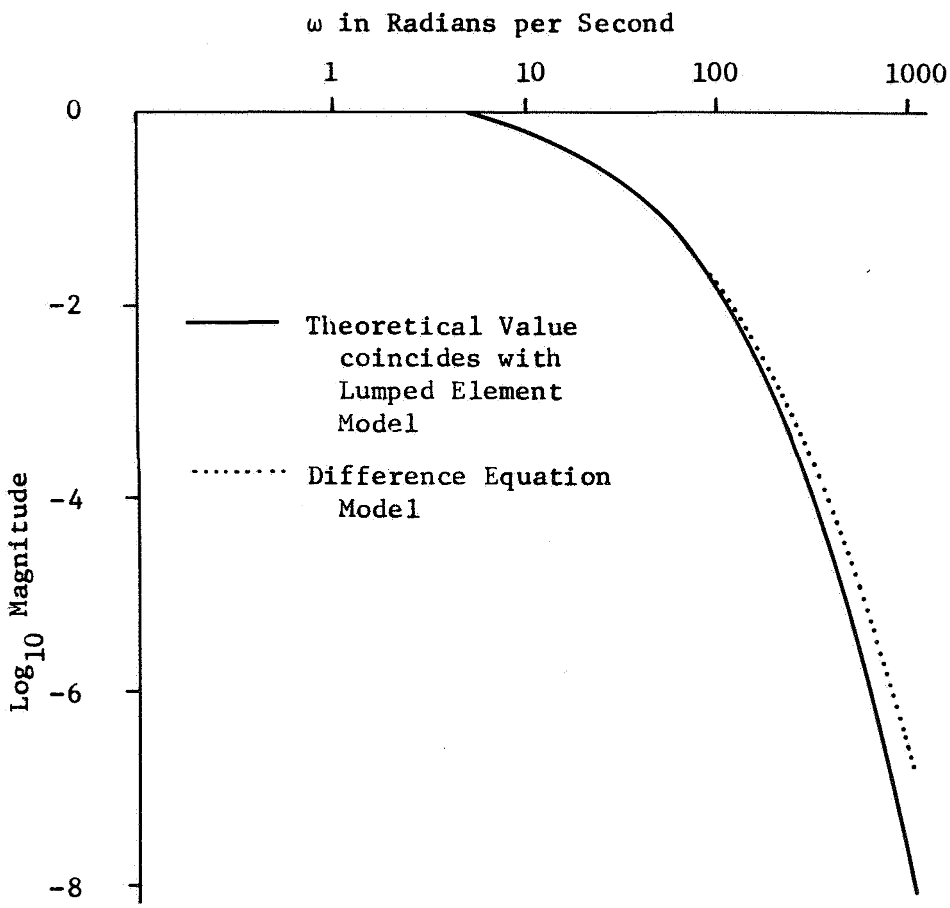

Fig. 3.2 Magnitude of the y_{12} Paramter

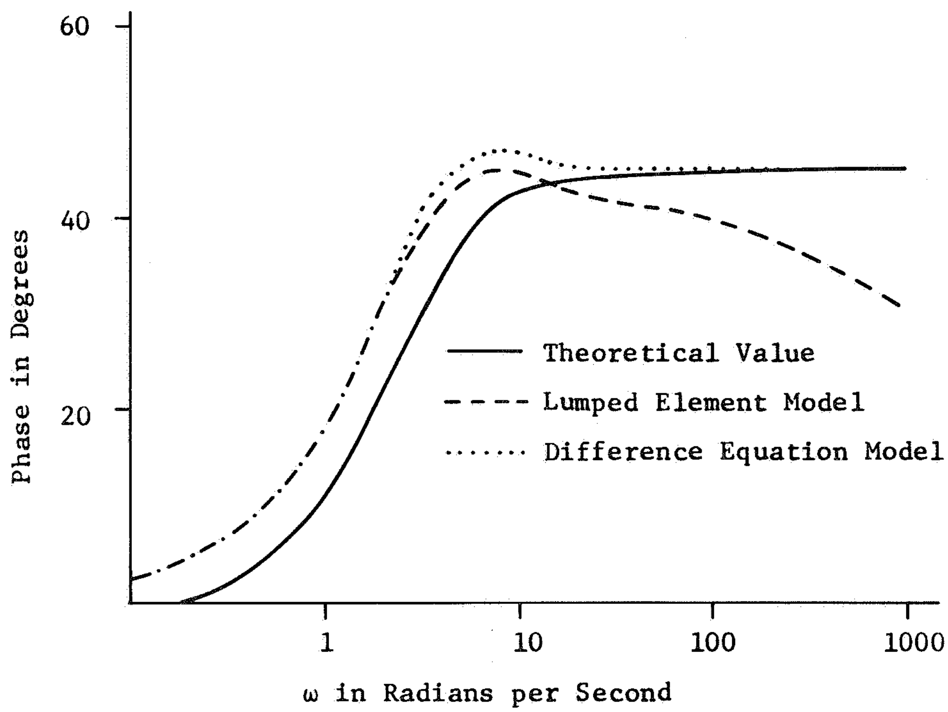

Fig. 3.3 Phase of the y_{11} Parameter

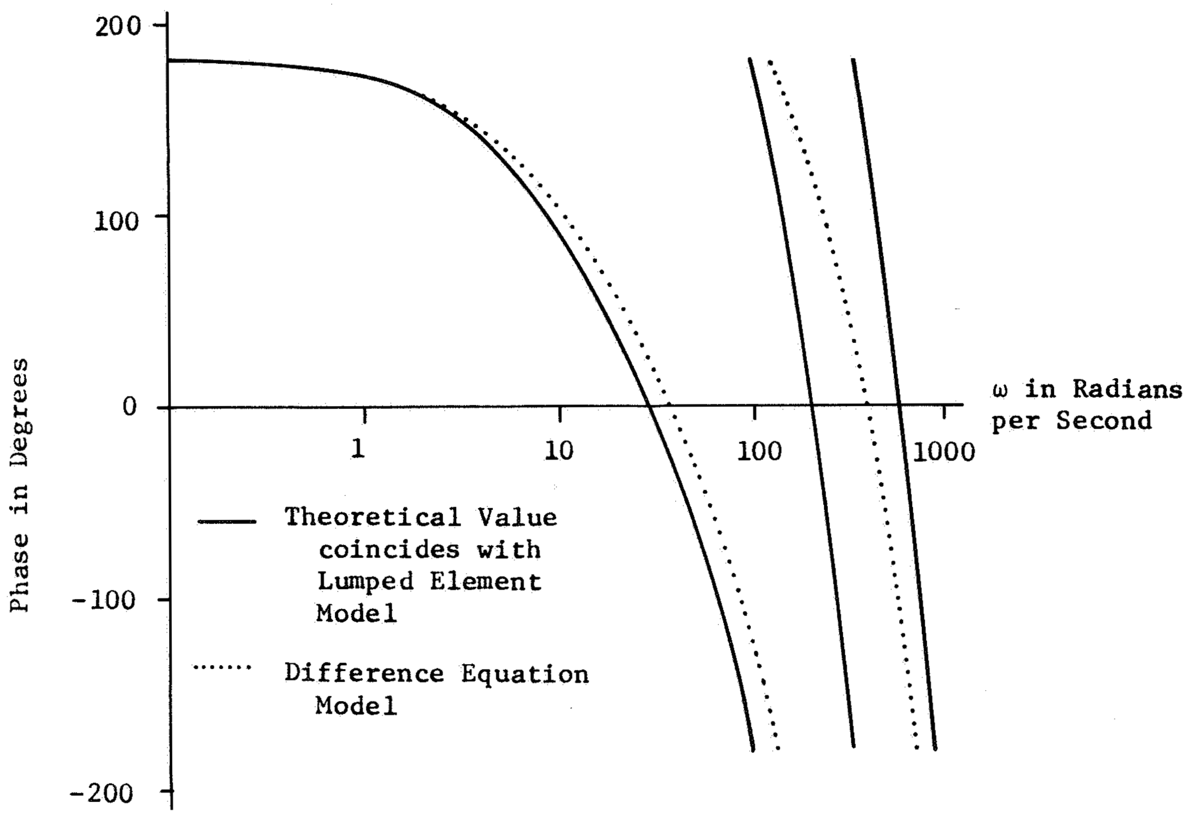

Fig. 3.4 Phase of the y_{12} Parameter

Equation Model. The magnitudes of the error were of the same order as for the y_{21} and y_{11} parameters, so additional graphs were not included. As a general rule, then, the Difference Equation Model gives better results for the input and output admittance parameters and the Lumped Element Model gives better results for the transfer admittance parameters.

The graphs of Fig. 3.1 through Fig. 3.4 also show that there is a much larger error in both the phases and magnitude using the Difference Equation Model for the transfer parameters than there is using the Lumped Element Model for the input parameters. This would seem to indicate that the Lumped Element Model would provide better overall accuracy for the entire network. That this is indeed the case will be shown in the following sections.

Comparison of Average Error vs. Frequency

The graphs of Fig. 3.5 and Fig. 3.6 show the average error in the magnitude and phase respectively for both models for various values of ω and for various numbers of elemental sections. By average error, we mean the magnitudes of the errors in the four parameters were added and normalized by dividing by four.

The graphs show clearly that the error in both magnitude and phase increases rapidly at high frequencies. A comparison shows that the error in both the magnitude and phase is smaller for the Lumped Element Model when the same number of sections is used. As expected, the graphs show that the error decreases for both models as the number of sections is increased.

Notice that the average error in magnitude for the Lumped Element Model using 40 sections is nearly as low as the error of the Difference Equation Model at all frequencies, and the average error in phase for the Lumped Element Model using 20 sections is lower than the error for the Difference Equation Model using 100 sections at any frequency above 20 radians per second.

In summary, the graphs of Fig. 3.5 and Fig. 3.6 show that the

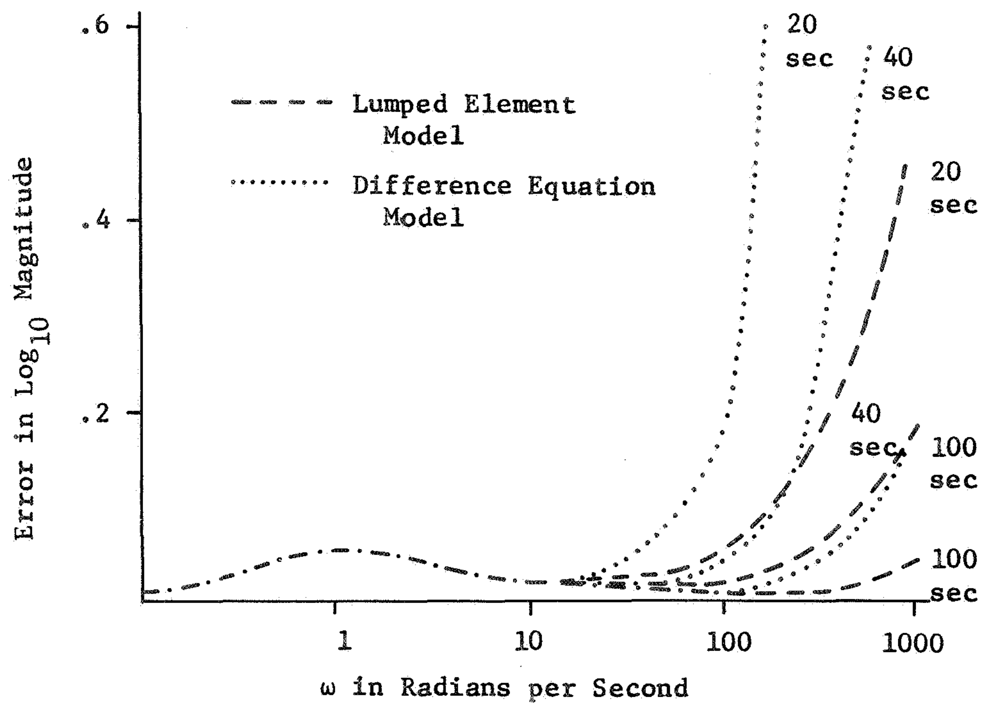

Fig. 3.5 Average Parameter Magnitude Error as a Function of Frequency for Various Numbers of Sections

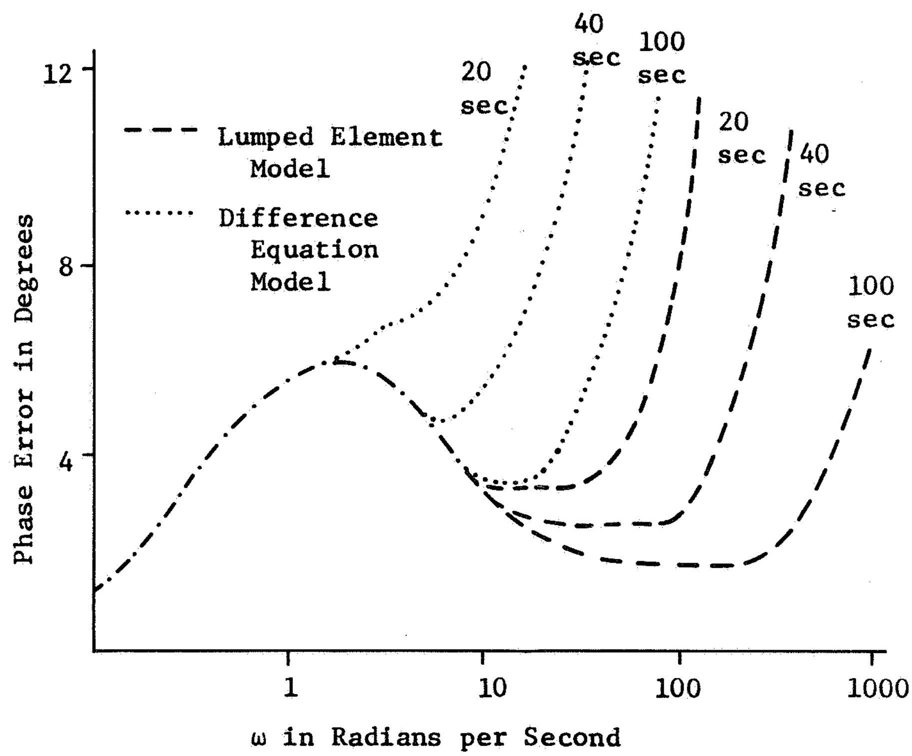

Fig. 3.6 Average Parameter Phase Error as a Function of Frequency for Various Numbers of Sections

Lumped Element Model provides a much more accurate overall approximation to the admittance parameters at high frequencies.

Introduction of an Overall Performance Criterion

Although the graphs in Fig. 3.1 through Fig. 3.6 indicate that the error in approximation is lower using the Lumped Element Model at high frequencies, they fail to give a meaningful representation of the error we might expect in a typical application of the distributed network, or of the additional benefits as the number of sections used in the models is increased. An overall performance criterion that indicates the expected accuracy of the approximation of a network in a typical circuit application was therefore developed.

Kaufman and Garrett have shown that when a uniform distributed network is used as a notch filter, the frequency of the notch is about 10 radians per second if unity values of R and C are used.[1] The performance criterion developed assumes that in most circuit applications, the frequencies of interest will range from 1 to 100 radians per second (assuming R and C are normalized to unity), and that in most cases the frequency range of interest will center around 10 radians per second.

The overall performance criterion gives the most probable average error in an admittance parameter if the most probable frequencies of interest are normally distributed logarithmically about 10 radians per second. That is, the average error at any frequency is multiplied by the density function $\theta(\omega)$, and multiplied by the interval between frequency sample points Δ, where the quantities $\phi(\omega)$ and Δ are defined by the following relations:

$$\phi(\omega) = \emptyset(\log\,(\omega/10)) \qquad (3.5)$$

where $$\emptyset(x) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{x^2}{2}} \qquad (3.6)$$

$$\Delta = \log \omega_2 - \log \omega_1 \qquad (3.7)$$

where ω_1 and ω_2 are successive frequency sample points, ω_2 is greater than ω_1, and ω is the geometric mean of ω_2 and ω_1. The function $\emptyset(x)$ is the normal distribution function centered at zero with variance equal to one.[3]

The overall performance criterion is not intended primarily to indicate exactly the numberical value of the most probable average error in an admittance parameter, but rather to provide some indication of the order of error that one might expect in a typical application, and to provide a meaning basis for comparison of the two models. The criterion is also used to show improvement in performance as the number of sections used is increased.

Comparison of the Models on the Basis of the Overall Performance Criterion

The graphs of Fig. 3.7 and Fig. 3.8 show the average error to be expected respectively in magnitude and phase for the two models using various numbers of sections. The average error to be expected is calculated on the basis of the overall performance criterion previously defined. The graphs show that the overall performance of the Difference Equation Model approaches that of the Lumped Element Model with respect to magnitude error as the number of sections is increased to 100. On the other hand, the overall performance with respect to phase error is much better using the Lumped Element Model, even as the number of sections is increased to 100. These two graphs provide the final evidence needed to declare that the Lumped Element Model gives the better overall performance of the two for the case of the uniform distributed network.

The graphs also show that, for the case of the uniform distributed network, the average error to be expected does not decrease significantly with an increasing number of sections after about 60 sections. This would indicate that most of the improvement in accuracy occurs at frequencies above 100 radians per second after the number of sections is increased above 60.

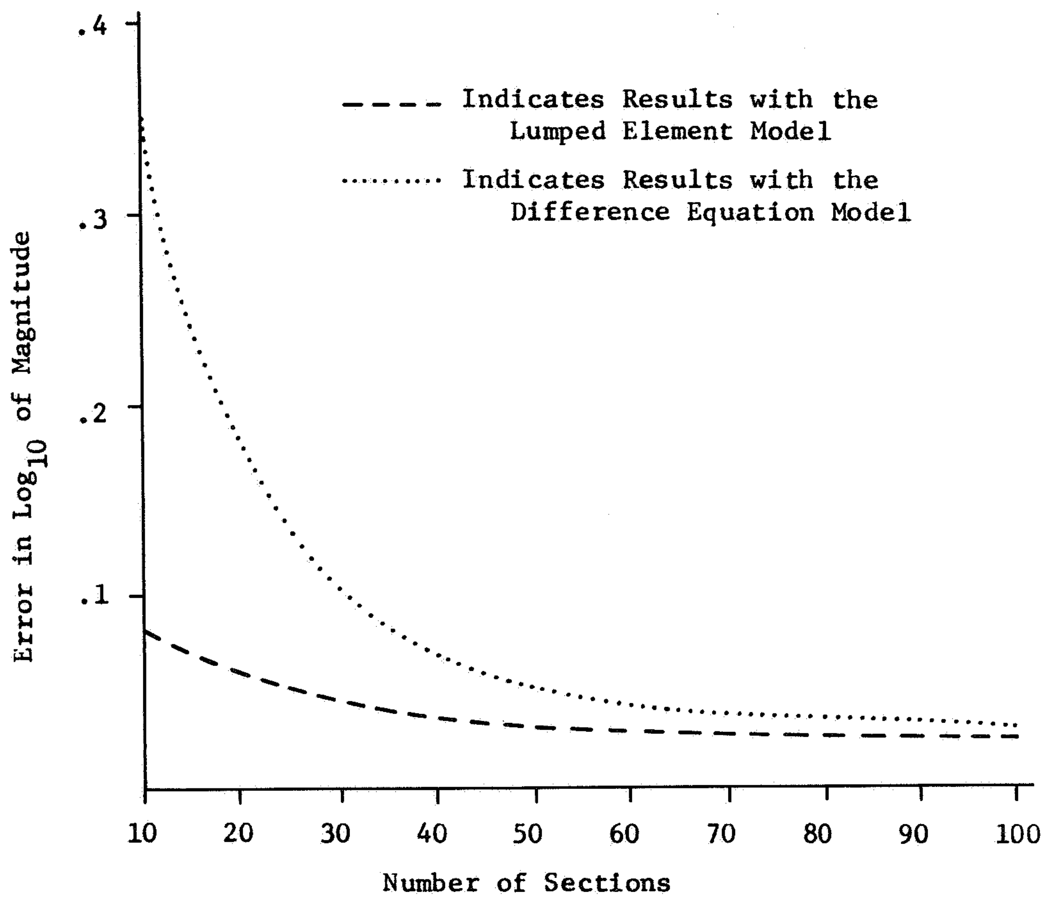

Fig. 3.7 Typical Parameter Magnitude Error to be Expected in Application of a Uniform Distributed Network

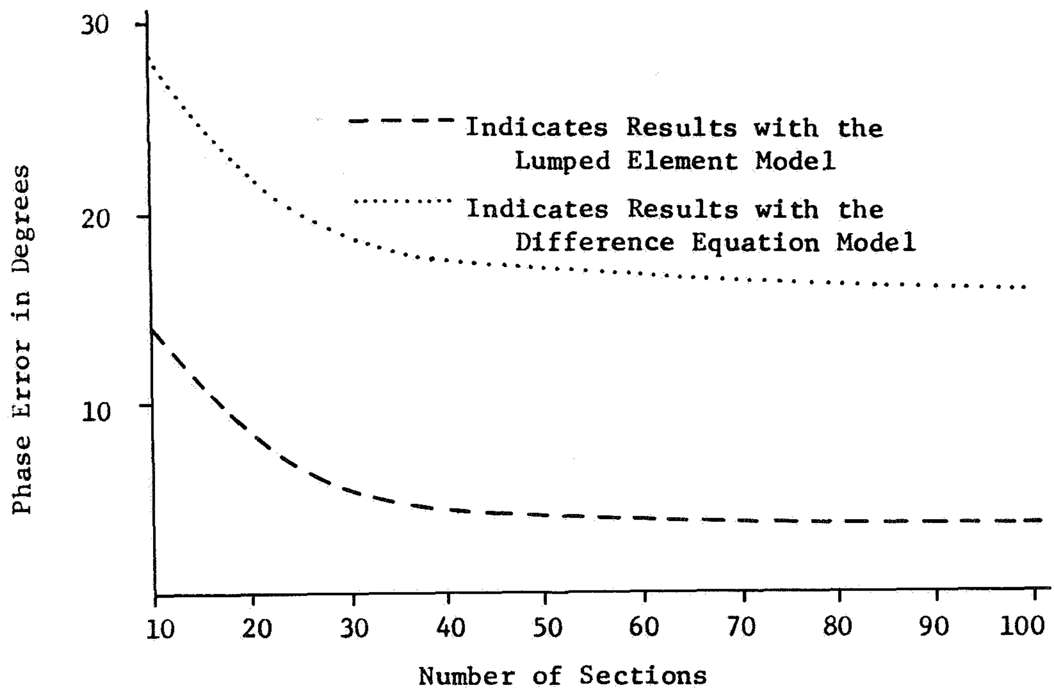

Fig. 3.8 Typical Parameter Phase Error to be Expected in Application of a Uniform Distributed Network

From the above discussion we see that when the results for the admittance parameters of the uniform distributed network as obtained from the two models are compared with theoretical results, certain patterns are present in the approximation error. The Difference Equation Model gives better results for input and output admittance parameters, while the Lumped Element Model gives better results for the transfer parameters. For both models, error increases rapidly at high frequency, and the average error in both magnitude and phase is less at higher frequencies using the Lumped Element Model. The expected average error, calculated using an overall performance criterion, does not decrease significantly after the number of sections is increased past about 60 for both models, but a comparison of this error provides a final indication that the Lumped Element Model has a greater degree of overall accuracy.

IV. RESULTS WITH AN EXPONENTIALLY TAPERED DISTRIBUTED NETWORK

In this section, we compare the results for the admittance parameters of the exponentially tapered network obtained from the models with the theoretical values. The overall performance of the models is compared on the basis of the overall performance criterion introduced in the last section.

Results Comparing the Accuracy of the Parameters

A computer program was developed to compare the results for the admittance parameters from (3.3) with the results obtained from both models. The results were compared for $\alpha = 1$, $\alpha = 2$, and $\alpha = 3$. In all cases, the conclusions reached were the same as in the case of the uniform distributed network. Both models approximate the results to the same degree of accuracy at low frequencies, but the Difference Equation Model gives better results for y_{11} and y_{22} at high frequencies. The Lumped Element Model gives better results for the y_{12} and y_{21} parameters at high frequencies. As with the uniform distributed network, the average error in phase and magnitude both increase rapidly with increasing frequency, but if both models use the same number of elemental sections, the Lumped Element Model provides a more accurate approximation at high frequency.

Comparison of the Models with Respect to Various Numbers of Sections using the Overall Performance Criterion

Kaufman and Garrett have shown that when the exponentially tapered distributed network is used as a notch filter, the frequency of the notch is about 10 radians per second if unity values of r_o, c_o, and L are used.[1] The frequency is nearly the same for α equal to zero, one, two, and three. We may then use the overall performance criterion, introduced in the last section, for the exponentially tapered network as well as for the uniform network.

The graphs of Fig. 4.1 and Fig. 4.2 show the average expected error in magnitude and phase respectively for various numbers of sections and for various values of α. The graphs of Fig. 3.7 and Fig. 3.8 for the uniform network ($\alpha = 0$) are shown on the same scales for reference. The graph of Fig. 4.1 shows that the expected average error in magnitude decreases uniformly with increasing number of sections for both models and for all values of α. The error in the Difference Equation Model is virtually the same for α equal to one, two, or three, and runs slightly less than the error for α equal to zero. In the case of the Lumped Element Model, the error for low numbers of sections increases as α increases from one to three, and the error is again less than for α equal to zero for more than 30 sections.

The expected average error in phase shown in the graph of Fig. 4.2 shows that for the Difference Equation Model, the phase error decreases uniformly as the number of sections is increased for all values of α, and the graphs for α equal to one, two, and three coincide as in the case of the magnitude error. The apparently strange behavior in the graphs for the Lumped Element Model requires an explanation. One of the frequencies used for comparison of the models was 100 radians per second. As in the case of the uniform network, there is a change in

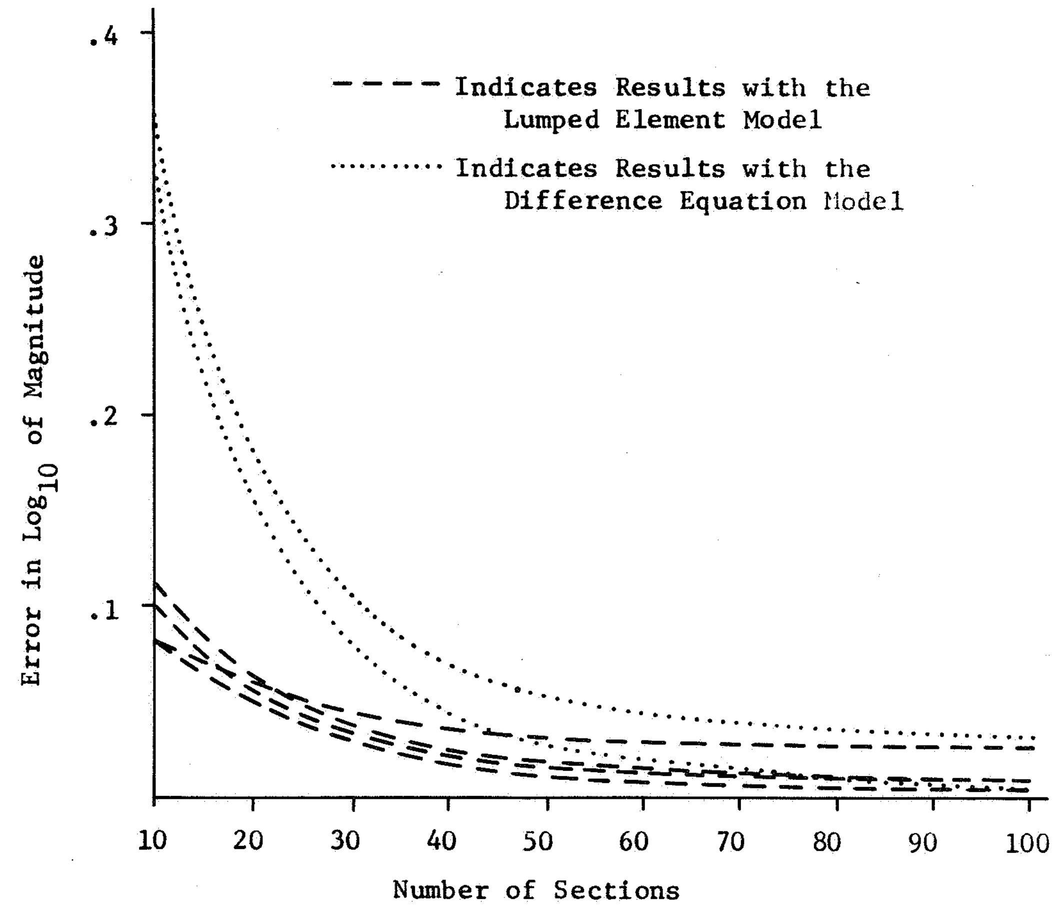

Fig. 4.1 Typical Parameter Magnitude Error to be Expected in Application of an Exponentially Tapered Distributed Network

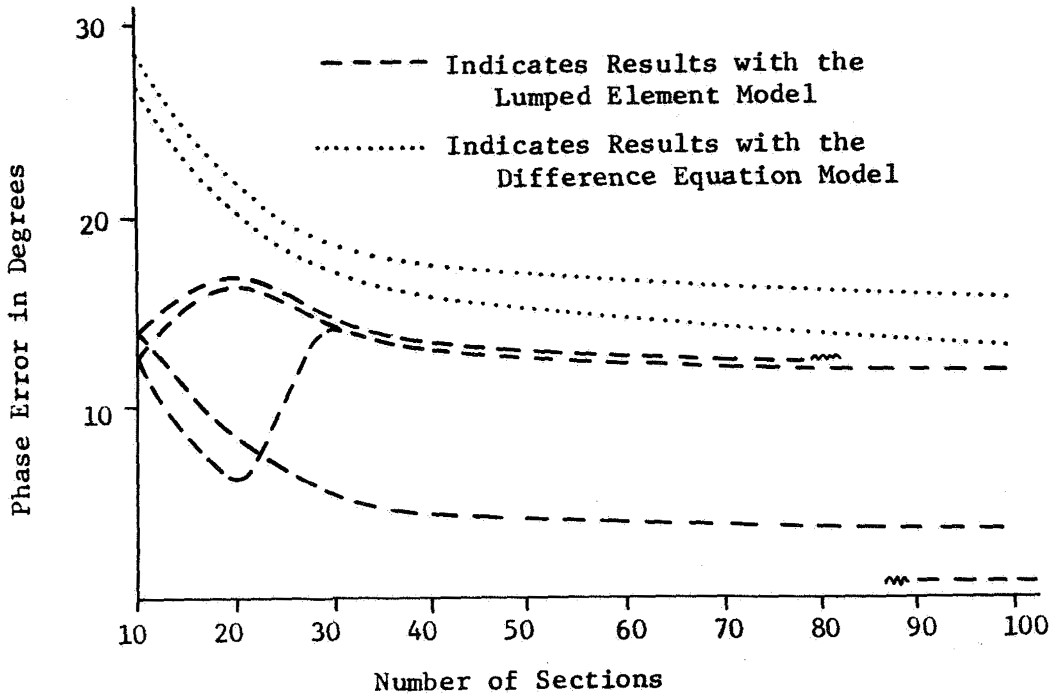

Fig. 4.2 Typical Parameter Phase Error to be Expected in Application of an Exponentially Tapered Distributed Network

the phase of the y_{12} and y_{21} parameters from a value of -180 degrees to +180 degrees near 100 radians per second. This effective change in the value of phase for the uniform network is shown in the graph of Fig. 3.4. This change in phase resulted in nominal phase errors of nearly 360 degrees at the frequency of 100 radians per second for most cases using the Lumped Element Model. This large error at 100 radians per second, which is the direct result of defining the phase in such a way as to make it a single valued function of frequency, causes the high phase error for the exponentially tapered distributed network as shown in the graphs. The low phase error for 100 sections when α equals three provides a more accurate indication of the performance of the Lumped Element Model. The phase error then, is actually even less for the Lumped Element Model than might be apparent from consideration of the graph alone. We may then conclude that the Lumped Element Model gives better overall results than the Difference Equation Model for an exponentially tapered network.

V. CONCLUSION

The results as presented in the preceding sections of this report show that both models give an accurate approximation to the admittance parameters of a uniform or exponentially tapered distributed network. Though only these two special cases were tested, the techniques are readily applicable to determine an approximation for any three terminal distributed RC network.

In practice, the use of the Lumped Element Model will entail little additional computation time. This follows from the fact that most of the computation involves the determination of R_k and C_k for each section of the tapered distributed network and the multiplication of the matrices. Thus, the use of the Lumped Element Model involves only a slight increase in computation time. To see this note that a single multiplication is required to form the elements of the matrix of (2.5), while two multiplications and one addition are required to form the elements of the matrix of (2.5), while two multiplications and one addition are required to form the elements of the matrix of (2.10). Thus, if there are n sections, setting up the n matrices required for the Lumped Element Model requires 3n operations, while the same process requires n operations in the case of the Difference Equation Model. Now consider the operations necessary for multiplying the matrices. These will be the same no matter which model is used. Each matrix multiplication requires eight complex multiplications and four complex additions. Each complex addition requires two operations, and each complex multiplication requires six operations so the matrix multiplication will require a total of 56(n-1) operations. Clearly, the extra time required by the 2n additional operations necessary to set up the matrices for the Lumped Element Model will be negligible compared to the total. Thus we conclude that the model proposed by E. C. Bertinolli which is equivalent to our Difference Equation Model may be considerably improved with an insignificant increase in computation time. Evidently neglecting the effects of second order differential terms is unwise when a finite number of sections is used to approximate the distributed line.

The primary importance of the models analyzed in this report lies in the fact that they provide a technique that may be applied to any three terminal distributed network, regardless of the geometry of that network. Previous techniques of analysis depend upon the geometry of the network, and there are many geometries which remain unexplored for the reason that no analysis technique has been available. The models presented here may easily be incorporated in a network analysis computer program that could be used in the analysis of circuits which employ distributed networks in addition to lumped elements and active devices. Such a program has been developed and will be discussed in a subsequent report.

SELECTED BIBLIOGRAPHY

1. Kaufman, W. M. and S. J. Garrett. Tapered distributed filters, IRE Trans. on Circuit Theory, Vol CT-9, December, 1962, pp. 329-336.

2. Bertinolli, E. C. and C. A. Halijak, Distributed Parameter RC Network Analysis, IEEE International Convention Record, Part 7, March 1966, pp. 243-247.

3. Fisz, M. Probability Theory and Mathematical Statistics, Third Edition. John Wiley and Sons, Inc., New York, 1963.